A Simple Refutation of the Bell-CHSH Implications

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Abstract

We show the conditions represented by the Bell-CHSH inequality do not imply quantum mechanics is a nonlocal theory.

Keywords

Bell-CHSH inequality, Nonlocality, Local Realism, Hidden Variables, Local Factoring, Bell's Theorem

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1 – Local Realism

The Bell-CHSH inequality [1] purports to present simple mathematical limits that apply to all local realistic physical theories. Experiments have shown that quantum mechanics violates these limits [2] and the generally accepted conclusion is that this demonstrates that quantum mechanics is not a local realistic theory.

The definition of what's local and realistic is inherently confusing because while locality is well defined, reality is not. For a theory to be local simply means that its predictions only involve forces and values within the light cone of what is observed.

Realism, on the other hand, is defined as making a certain, definite prediction that pertains to the outcome of any observation anywhere. That is to say, a theory that makes such a prediction is realistic, and what it predicts is an element of reality according to that theory. If a theory cannot make such predictions about an observable quantity, or if it makes predictions that are not observed or observable, then that quantity is not "Einstein real," which is supposed to make it unreal along with that part of the theory that generated a definite value for it.

Here we look at Bell's local factoring assumption, essential in deriving the limiting condition known as the Bell-CHSH inequality. We show that it is an unnecessary condition by offering another construction that is structurally dissimilar but equally local.

We don't know what or if there is any comprehensive assumption with which to replace Bell's factoring assumption and we think that's beside the point. By simply replacing it with a slightly broader locality condition that reproduces the quantum mechanical result we show that a theory's locality is not determined by whether or not it adheres to the Bell-CHSH inequality.

We're not concerning ourselves with the question of realism, Einsteinian or otherwise, because it lacks definition and its effect in defining a theory is uncertain. What is presented here is not a physical argument or theory, but a mathematical demonstration that Bell's local factoring assumption can be broadened to retain locality and reproduce the quantum mechanical result. From this we can conclude Bell's local factoring is not a necessary condition, and that experimental confirmation of the quantum prediction has no bearing on the question of locality in physical theories. This result is argued in greater detail in a separate publication [3].

2 – The Binary Observation

Various quantum properties are observed only to be binary, this includes particle spin and polarization. Only two values are observed and the theory requires only a two-dimensional space to describe them. In addition, systems are quantized in these two dimensions so that only one of two values are observed.

Following Bell, we define A and B to be the result of an observation of one of these binary quantities at either of two locations A and B. We'll call the results +1 or -1 at either location.

3 – The Correlation Function

We consider a system of two particles that are correlated with opposite properties, spin or polarization, such that each particle has a property opposite to its partner: if one is spin +1 in a particular direction, then the other is spin -1 when measured in the same direction.

We then ask what happens when we measure the spins in different directions at each of the two locations. By the quantized, dual nature of spin, no matter what direction we choose to measure, we'll only get a +1 or -1 result.

How to reconcile this quantum nature with the requirement that the two spins are always diametrically opposite is a physics puzzle we don't address here. All we're doing is defining a local correlation function, E(a,b), which gives the probability that the two measured values will be the same or different as a function of two angles, a and b.

The angle a is the direction from the vertical in which spin is measured at A, and b is measured in the same plane as the direction from the vertical in which spin is measured at B. When a and b are the same, then the anti-correlation condition requires that the values measured at A and B will be opposite. When a and b are antiparallel, differing by an angle π, the condition requires the measured value will be the same. That is, if we measure A at zero degrees from the vertical and get the result +1, then if we measure B at 180-degrees from the vertical we'll also get the result +1 because +1 is really -1 measured upside down.

The correlation function tells us the probability of finding the two spin measurement outcomes to be opposite or equal as a function of the angles a and b. They'll be opposite 100% of the time if a and b are equal, and they'll be opposite 0% of the time if a=b+180-degrees. But if a and b differ by less than 180-degrees then the spins will be measured as opposite more than zero and less than 100% of the time.

Quantum mechanics gives us a closed form expression for $E(a,b) = -\cos(a-b)$ for any angles $0<|a-b|<\pi/2$. The expression derives from the way spin states, which obey an SU(2) algebra appropriate for these 2-dimensional objects, project onto each other when one is rotated with respect to the other.

In order to show that quantum mechanics is not a local, realistic theory the Bell-CHSH program proceeds to make a local factoring assumption that is asserted to hold for all local, realistic theories. This local factoring assumption written as:

$$E(a,b)=F(a)G(b) \tag{1}$$

The measurement outcomes at A and B are A(a) and B(b), each of which have been confirmed to take only the values +1 and -1.

4 – Auxiliary "Hidden" Variables

Following Bell and others, we introduce a set of auxiliary variables represented by $\{\lambda\}$. We'll consider $\boldsymbol{\lambda}$ to be a 3-vector that's involved in establishing this correlation. The components of $\boldsymbol{\lambda}$ are referred to as hidden variables because, it's said, they don't exist in quantum mechanics and have not been otherwise observed. We posit that $F(\mathbf{a})$ is influenced by $\boldsymbol{\lambda}$ and takes values that depend on it.

Let $f(\boldsymbol{\lambda})$ be the likelihood of $A(\mathbf{a},\boldsymbol{\lambda})$ being either +1 or -1. Without loss of generality we can write $F(\mathbf{a},\boldsymbol{\lambda})$ as a product of the various values of A weighted by the probability of their outcome. That is, for certain values of $\boldsymbol{\lambda}$ we can write

$$F(\mathbf{a},\boldsymbol{\lambda}) = f(\boldsymbol{\lambda})A(\mathbf{a},\boldsymbol{\lambda})$$

As a probability, $f(\boldsymbol{\lambda})$ always has a value between 0 and 1. It should also be normalized so that the sum of all its weighting values add up to 1. Assuming $\boldsymbol{\lambda}$ takes continuous values means its integral over all values of $\boldsymbol{\lambda}$ should be 1.

$$\int f(\boldsymbol{\lambda})\, d\boldsymbol{\lambda} = 1$$

However, for the purpose of fabricating a local correlation function we don't need to understand $f(\boldsymbol{\lambda})$ as a probability distribution. As we'll see while $f(\boldsymbol{\lambda})$ is local, it does not satisfy the probability requirement of integrating to 1.

We write the expectation value of $F(\mathbf{a})$ as the convolution of $f(\boldsymbol{\lambda})$ and $A(\mathbf{a},\boldsymbol{\lambda})$. Both $\mathbf{a}$ and $\boldsymbol{\lambda}$ are 3-dimensional vectors. This gives us:

$$F(\mathbf{a})=\int F(\mathbf{a},\boldsymbol{\lambda})\, d\boldsymbol{\lambda} = \int f(\boldsymbol{\lambda})A(\mathbf{a},\boldsymbol{\lambda})\, d\boldsymbol{\lambda} \tag{2}$$

Considering E(**a**,**b**) as a probability, it is equal to the sum of all appropriately weighted possible values. Each weighting, given by f(**λ**), corresponds to a configuration of A's values.

We similarly define a weight function g(**λ**) for each of the values of B(**b**,**λ**), with f(**λ**) and g(**λ**) being functions of the same 3-dimensional vector **λ**. Adding all possible configurations, we can represent the correlation probability E(**a**,**b**) as an integral over all **λ** as:

$$E(\mathbf{a},\mathbf{b})=\int f(\boldsymbol{\lambda})g(\boldsymbol{\lambda})\, A(\mathbf{a},\boldsymbol{\lambda})\, B(\mathbf{b},\boldsymbol{\lambda})\, d\boldsymbol{\lambda} \qquad (3)$$

5 – The Bell-CSHS Inequality

Bell [4] and Clauser, et al. [5] then consider the variable S defined as a combination of expectations at four different angles a, a', b, and b', given by four different vectors **a**, **a'**, **b**, and **b'** lying in the same plane, as:

$$S = E(\mathbf{a},\mathbf{b}) - E(\mathbf{a'},\mathbf{b}) + E(\mathbf{a},\mathbf{b'}) + E(\mathbf{a'},\mathbf{b'})$$

$$E(\mathbf{a},\mathbf{b})=\int f(\boldsymbol{\lambda})g(\boldsymbol{\lambda})\, [A(\mathbf{a},\boldsymbol{\lambda})\, B(\mathbf{b},\boldsymbol{\lambda}) - A(\mathbf{a'},\boldsymbol{\lambda})\, B(\mathbf{b},\boldsymbol{\lambda}) + A(\mathbf{a},\boldsymbol{\lambda})\, B(\mathbf{b'},\boldsymbol{\lambda}) + A(\mathbf{a'},\boldsymbol{\lambda})\, B(\mathbf{b'},\boldsymbol{\lambda})]\, d\boldsymbol{\lambda}$$

$$E(\mathbf{a},\mathbf{b})=\int f(\boldsymbol{\lambda})g(\boldsymbol{\lambda})\, [A(\mathbf{a},\boldsymbol{\lambda})\, B(\mathbf{b},\boldsymbol{\lambda}) + A(\mathbf{a},\boldsymbol{\lambda})\, B(\mathbf{b'},\boldsymbol{\lambda}) - A(\mathbf{a'},\boldsymbol{\lambda})\, B(\mathbf{b},\boldsymbol{\lambda}) + A(\mathbf{a'},\boldsymbol{\lambda})\, B(\mathbf{b'},\boldsymbol{\lambda})]d\boldsymbol{\lambda}$$

$$E(\mathbf{a},\mathbf{b})=\int f(\boldsymbol{\lambda})g(\boldsymbol{\lambda})\, \big[A(\mathbf{a},\boldsymbol{\lambda})\, \big(B(\mathbf{b},\boldsymbol{\lambda}) + B(\mathbf{b'},\boldsymbol{\lambda})\big) - A(\mathbf{a'},\boldsymbol{\lambda})\, \big(B(\mathbf{b},\boldsymbol{\lambda}) - B(\mathbf{b'},\boldsymbol{\lambda})\big)\big]d\boldsymbol{\lambda} \qquad (4)$$

For all values of **λ**, A(**a**,**λ**) = ±1 and B(**b**,**λ**) = ±1. By considering all combinations we can conclude:

$$\big[A(\mathbf{a},\boldsymbol{\lambda})\, \big(B(\mathbf{b},\boldsymbol{\lambda}) + B(\mathbf{b'},\boldsymbol{\lambda})\big) - A(\mathbf{a'},\boldsymbol{\lambda})\, \big(B(\mathbf{b},\boldsymbol{\lambda}) - B(\mathbf{b'},\boldsymbol{\lambda})\big)\big] = +/-2$$

Since $\int f(\boldsymbol{\lambda})\, g(\boldsymbol{\lambda})\, d\boldsymbol{\lambda} = 1$, that implies:

$$-2 <= S <= +2$$

However, using the quantum mechanical correlation cos(a–b), and substituting angle a = 0, b = π/8, a' = 2π/8, and b' = 3π/8, gives the value of S as:

$$S = \cos(a-b) - \cos(a'-b) + \cos(a-b') + \cos(a'-b')$$

$$S = \cos(-\pi/8) - \cos(\pi/8)) + \cos(-3\pi/8) + \cos(\pi/8)) = 2.828$$

From this it is concluded that quantum mechanics violates the bounds of ±2 that apply to the foregoing assumptions about local theories.

6 – An Alternative Local Factoring

Rather than assuming F(**a**,**λ**) = f(**λ**)A(**a**,**λ**), let us assume instead that the weighting function f(**λ**) is also a function of the angle at which the measurement is made. Here **a** and **λ** are 3-vectors whose values define angles a and λ measured in a plane shared by **a** and **λ**. The length of **a** is 1, and the length of **λ** is K. We now have:

$$F(\mathbf{a},\boldsymbol{\lambda}) = f(\mathbf{a},\boldsymbol{\lambda})A(\mathbf{a},\boldsymbol{\lambda})$$

Consider the weighting function to be:

$$f(\mathbf{a},\boldsymbol{\lambda}) = |\mathbf{a}\cdot\boldsymbol{\lambda}| = K\,|\cos(a-\lambda)|$$

a and **λ** are vectors both of whose bases are at the origin of a sphere in 3-dimensions and whose heads lie anywhere on the surface of S2, the sphere in 3-dimensions. The sum of f(**a**,**λ**)'s values is the integral of **λ** over S2, which in spherical coordinates is the integral over the polar angle θ from 0 to π, and the azimuthal angle ϕ from 0 to 2π. Because the lengths of **a** and **λ** are constant, and the inner product of two vectors is independent of direction and is rotationally symmetric around an axis defined by the vector **a**, the integral of f(**a**,**λ**) is a constant that does not depend on **a**.

$$\int f(\mathbf{a},\boldsymbol{\lambda})\, d\boldsymbol{\lambda} = K \int |\cos(a-\lambda)|\, d\boldsymbol{\lambda}$$

By choosing a coordinate system whose polar direction is coincident with **a**, the integral of the polar angle θ that's made between **a** and **λ** ranges from θ = 0 to π.

$$\int f(\mathbf{a},\boldsymbol{\lambda})\, d\boldsymbol{\lambda} = K \int_{\theta=1}^{\pi} \int_{\phi=0}^{2\pi} |\cos(\theta)|\, \sin(\theta)d\theta\, d\phi$$

$$= 2K \int_{\theta=1}^{\pi/2} \cos(\theta)\, \sin(\theta)d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= 2K\, (½)\, (2\pi)$$

$$= 2\pi K$$

If we chose K = (1/2π) then the integral of f(**a**,**λ**) will equal one, and we can interpret f(**a**,**λ**) as a probability. However, we need to leave K as a free parameter in order to match our final result with the correct quantum correlation. This being the case, we're left to interpret f(**a**,**λ**) not as a probability, but as a scaling function that contains some element of the system's dynamics.

It's worth noting that f(**a**,**λ**)'s proportionality to 2π comes from the azimuthal integral. This azimuthal symmetry is a fabrication of our mapping of the 2-dimensional nature of spin onto the 3-dimensions of the experiment. The quantized SU(2) system we're modeling, whether it be spin or polarization, has only two orthogonal bases, and only one parameter is required to rotate between them. However, here we're coupling the measurement vectors **a** and **b** to the 3-dimensional "hidden variable" **λ**, and we must integrate over all values of **λ** in order to eliminate **λ** from the final result. The 2π factor is generated by this second integration that's necessary to obtain a result in 3-dimensions but superfluous in describing the probabilities of spin orientations in 2-dimensions.

7 – A Local Correlation Function

We can write $A(\mathbf{a},\boldsymbol{\lambda}) = \text{sign}(\mathbf{a}\cdot\boldsymbol{\lambda}) = \pm 1$, and assume that at site B the vector $\boldsymbol{\lambda}$ points in the opposite direction from the vector $\boldsymbol{\lambda}$ at site A. That is, $B(\mathbf{b},\boldsymbol{\lambda}) = \text{sign}(\mathbf{b}\cdot -\boldsymbol{\lambda}) = -\text{sign}(\mathbf{b}\cdot\boldsymbol{\lambda})$. With this, A and B fully model the observed binary values of A and B for any value of $\boldsymbol{\lambda}$. That is, A and B are always ± 1, and always give opposite values when $\mathbf{a} = \mathbf{b}$, independent of $\boldsymbol{\lambda}$. We'll see that this gives the observed correlation values as well.

We then can write $F(\mathbf{a},\boldsymbol{\lambda})$ and $G(\mathbf{b},\boldsymbol{\lambda})$ as:

$$F(\mathbf{a},\boldsymbol{\lambda}) = (1/2\pi)|\mathbf{a}\cdot\boldsymbol{\lambda}|\, \text{sign}(\mathbf{a}\cdot\boldsymbol{\lambda}) = (1/2\pi)\, (\mathbf{a}\cdot\boldsymbol{\lambda}),$$

$$G(\mathbf{b},\boldsymbol{\lambda}) = (1/2\pi)|\mathbf{b}\cdot\boldsymbol{\lambda}|\, \text{sign}(-\mathbf{b}\cdot\boldsymbol{\lambda}) = - (1/2\pi)\, (\mathbf{b}\cdot\boldsymbol{\lambda})$$

The expectation value continues to be the product of two local terms:

$$E(\mathbf{a},\mathbf{b}) = \int F(\mathbf{a},\boldsymbol{\lambda})\, G(\mathbf{b},\boldsymbol{\lambda})\, d\boldsymbol{\lambda}$$

$$= - (1/2\pi)^2 \int (\mathbf{a}\cdot\boldsymbol{\lambda})(\mathbf{b}\cdot\boldsymbol{\lambda})\, d\boldsymbol{\lambda}$$

Writing the unit vector $\mathbf{a}$, of length one, and the vector $\boldsymbol{\lambda}$ of fixed length K as 3-vectors using polar coordinates gives: $\mathbf{a} = (a_1, a_2, a_3)$, and $\boldsymbol{\lambda} = (K\cos\phi \sin\theta, K\sin\phi \sin\theta, K\cos\theta)$. The integrals become:

$$= - (K/2\pi)^2 \int_{\theta=1}^{\pi} \int_{\phi=0}^{2\pi} \Big[(a_1, a_2, a_3)\cdot(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$$

$$\times \quad (b_1, b_2, b_3)\cdot(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)\Big] \sin\theta\, d\theta\, d\phi$$

$$= - (K/2\pi)^2 \int_{\theta=1}^{\pi} \Big(a_1 b_1 \sin^3\theta \left(\int_{\phi=0}^{2\pi} \cos^2\phi\, d\phi\right) + a_2 b_2 \sin^3\theta \left(\int_{\phi=0}^{2\pi} \sin^2\phi\, d\phi\right) +$$
$$+ a_3 b_3 \cos^2\theta \sin\theta \left(\int_{\phi=0}^{2\pi} d\phi\right)\Big)\, d\theta$$

These definite integrals are easily solved to yield:

$$= - (K/2\pi)^2\, (4\pi/3)\, \mathbf{a}\cdot\mathbf{b}$$

Setting $K = \sqrt{(3\pi)}$ reproduces the standard quantum correlation function which matches the observed correlations [6] predicted by quantum mechanics.

$$E(a,b) = -\, \mathbf{a}\cdot\mathbf{b} = -\cos(a-b)$$

The crucial point is this: a combination of local interactions can result in a correlation across any interval, as there are no conditions on the time or distance between regions A and B. A correlation carried by a local intermediary process remains a local effect.

We have reproduced the quantum results without adding any observable parameters or nonlocal interactions. This shows that experiments that confirm this cosine correlation can be obtained from a local quantum theory.

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